Dan Shanks' CUFFQI Algorithm Resurrected

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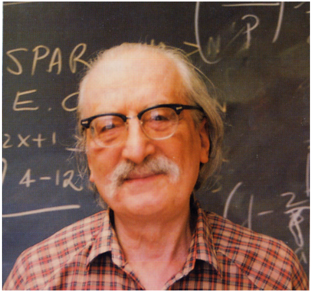

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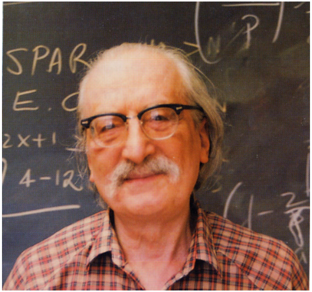

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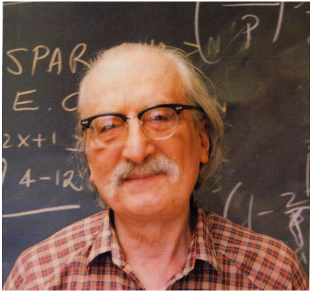
- Unpublished (to appear as Chapter 4 in *Cubic Fields With Geometry* by S. Hambleton & H. C. Williams, Springer Monograph 2018/19)



Cubic Fields

A cubic field of discriminant D has a generating polynomials of the form

$$f(x) = x^3 - 3N(\lambda)^{1/3}x + Tr(\lambda)$$

- λ is an algebraic integer in $\mathbb{Q}(\sqrt{-3D})$
- Norm and trace are taken in $\mathbb{Q}(\sqrt{-3D})/\mathbb{Q}$
- $N(\lambda) \in \mathbb{Z}^3$

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Roots of $f(x)$ (Cardano 1545):

$$\zeta^i \lambda^{1/3} + \zeta^{-i}\overline{\lambda}^{1/3} \qquad (i = 0, 1, 2)$$

where ζ is a primitive cube root of unity

Example: $D = 44806173$

Naively $\big($take λ to be the fundamental unit of $\mathbb{Q}(\sqrt{-3 \cdot 44806173})$ $\big)$:

$$f(x) = x^3 - 3x + 963135381187786734040565836 6$$

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Using CUFFQI (all 13 cubic fields with $D = 44806173$):

$$\begin{aligned}
f_1(x) &= x^3 - 61x^2 + 697x - 330 \\
f_2(x) &= x^3 - 279x^2 + 441x - 170 \\
f_3(x) &= x^3 - 63x^2 + 423x - 8 \\
f_4(x) &= x^3 - 69x^2 + 435x - 216 \\
f_5(x) &= x^3 - 63x^2 + 603x - 494 \\
f_6(x) &= x^3 - 83x^2 + 297x - 54 \\
f_7(x) &= x^3 - 63x^2 + 837x - 494 \\
f_8(x) &= x^3 - 257x^2 + 477x - 216 \\
f_9(x) &= x^3 - 87x^2 + 273x - 36 \\
f_{10}(x) &= x^3 - 62x^2 + 546x - 261 \\
f_{11}(x) &= x^3 - 60x^2 + 660x - 97 \\
f_{12}(x) &= x^3 - 165x^2 + 273x - 90 \\
f_{13}(x) &= x^3 - 127x^2 + 185x - 62
\end{aligned}$$

Cubic Field Construction

Problem with Berwick construction: polynomial coefficients can be HUGE!
$\Big($E.g. $Tr(\varepsilon) \approx \varepsilon \approx \exp(\sqrt{|D|})$ for the fundamental unit $\varepsilon \in \mathbb{Q}(\sqrt{-3D})\Big)$

CUFFQI to the rescue!

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CUFFQI to the rescue!

Goal: for a a given fundamental discriminant D, produce all the cubic fields of discriminant D à la Berwick via generating polynomials with small coefficients

The Berwick Map

There is a map from the set of unordered triples of conjugate cubic fields

$$\{ \mathbb{K}, \mathbb{K}', \mathbb{K}'' \} \qquad \text{disc}(\mathbb{K}) = D$$

to the set of unordered pairs of 3-torsion ideal classes

$$\{ [\mathfrak{a}], [\overline{\mathfrak{a}}] \}$$

in $\mathcal{O}_{\mathbb{Q}(\sqrt{-3D})}$ via

$$x^3 - 3N(\lambda)^{1/3}x + Tr(\lambda) \quad \longmapsto \quad \{ [\mathfrak{a}], [\overline{\mathfrak{a}}] \} \qquad \text{where } \mathfrak{a}^3 = (\lambda)$$

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For $D > 0$:

bijection onto non-principal ideal classes
nothing maps to the principal class

For $D < 0$:

3-to-1 onto non-principal ideal classes
1-to-1 onto to the principal class

Some Counting

Put

$$r = 3\text{-rank}\big(\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))\big)$$

$$s = 3\text{-rank}\big(\mathrm{Cl}(\mathbb{Q}(\sqrt{-3D}))\big)$$

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Number of cubic fields of discriminant D (Hasse 1929): $\frac{3^r - 1}{2}$

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Connection between r and s (Scholz 1932):

$|r - s| \leq 1$

If $r \neq s$, then the imaginary quadratic field has the bigger 3-rank

Case $D > 0$:

$r = s$: $$\frac{3^s - 1}{2} = \frac{3^r - 1}{2}$$ ☺

$r = s - 1$: $$\frac{3^s - 1}{2} = \frac{3^r - 1}{2} + 3^r$$ ☹

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So what are these extra 3^r cubic fields?

Answer: they are the complete collection of cubic fields of discriminant

$$9D \quad \text{if} \quad 3 \mid D, \qquad 81D \quad \text{if} \quad 3 \nmid D$$

In the ☺ cases there are no fields of these discriminants

Berwick Construction Algorithm

Input: D and a basis of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-3D})[3]$
$\big($For $D < 0$, also the regulator R of $\mathbb{Q}(\sqrt{-3D})\big)$

Output: generating polynomials of all cubic fields of discriminant D

Algorithm:

For each basis class $\mathcal{C}$ of $\mathrm{Cl}(\mathbb{Q}(\sqrt{-3D})[3]$, collect generators λ of

one ideal in $\mathcal{C}$ whose cube has a small generator when $D > 0$

three ideals in $\mathcal{C}$ whose cube has a small generator when $D < 0$

Collect a small element λ ($\notin \mathbb{Z}$) in some principal ideal when $D < 0$

For each λ collected

compute $f(x) = x^3 - 3N(\lambda)^{1/3}x + Tr(\lambda)$

if $\mathrm{disc}(f) = D$, output $f(x)$

Reduced Ideals

An ideal $\mathfrak{a}$ in $\mathcal{O}_{\mathbb{Q}(\sqrt{-3D})}$ is **reduced** if no non-zero element $\alpha \in \mathfrak{a}$ satisfies

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If $\mathfrak{a}$ is reduced, then

$$N(\mathfrak{a}) < \begin{cases} \sqrt{|D'|/3} & \text{when } D' < 0 \\ \sqrt{D'} & \text{when } D' > 0 \end{cases}$$

where $D' = -D/3$ when $3 \mid D$ and $D' = -3D$ when $3 \nmid D$.

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If $\mathfrak{a}$ is reduced and $\mathfrak{a}^3 = (\lambda)$, then

$$N(\lambda) < \begin{cases} (|D'|/3)^{3/2} & \text{when } D' < 0 \\ (D')^{3/2} & \text{when } D' > 0 \end{cases}$$

Hence, to get λ of small norm, use reduced ideals (exist in every ideal class)

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Here, the reduced ideal $\mathfrak{a}$ is unique.

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Happily, the reduced ideal also yields a small trace!

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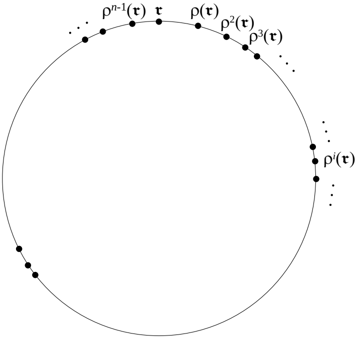
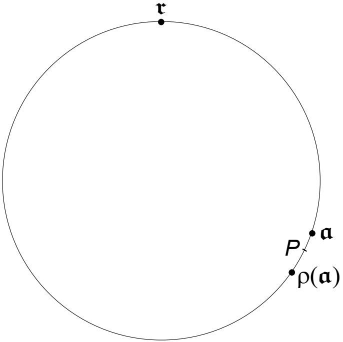
- Infrastructures are finite.
- Can move from one infrastructure ideal $\mathfrak{a}$ to its *neighbour* $\rho(\mathfrak{a})$ via one step in a simple continued fraction expansion
- Infrastructure ideals are discretely spaced on a circle of circumference R, the regulator of $\mathbb{Q}(\sqrt{D'})$
- For any point P on the circle, there is a unique reduced ideal *closest* to P (efficiently computable)

Infrastructures, $D' > 0$



Infrastructure of $\mathcal{C} = [\mathfrak{r}]$



$\mathfrak{a}$ is closest to P

Suitable Reduced Ideals, $D' < 0$

$\lambda \in \mathcal{O}_{\mathbb{Q}(\sqrt{D'})}$ is **small** if

$$1 < \lambda < (D')^{3/2}, \qquad |N(\lambda)| < (D')^{3/2}$$

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The following reduced ideals have cubes with small generators (Shanks):

- For the principal ideal class, the reduced ideal closest to

$$\frac{R}{3} + \frac{\log(D')}{4}$$

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- For the principal ideal class, the reduced ideal closest to

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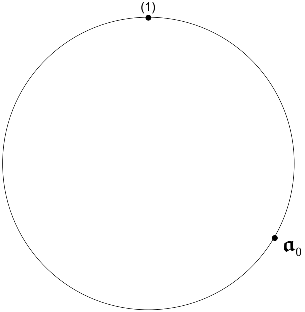
- For any non-principal ideal class $\mathcal{C}$, the three reduced ideals closest to

$$d\,, \quad \frac{R}{3} + d\,, \quad \frac{2R}{3} + d$$

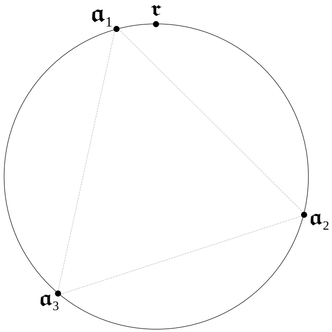
  where $0 < d < R/3$ and z can be explicitly computed

  (z depends on the representative of $\mathcal{C}$)

Suitable Reduced Ideals, $D' < 0$



Principal infrastructure



Non-principal infrastructures

Finding Small Generators, $D' > 0$

Shanks' strategy for finding λ (or $\overline{\lambda}$):

- Search the infrastructures of $[\mathfrak{a}]$ and of $[\,\overline{\mathfrak{a}}\,]$ simultaneously to find λ or $\overline{\lambda}$
- The two infrastructures are mirror images of each other

Fung's Work

In his 1990 PhD dissertation, Fung

- translated CUFFQI from Shanksian into a form suitable for computation
- implemented CUFFQI in Fortran on an Amdahl 5870 mainframe computer
- produced a number of examples, including the

$$\frac{3^6 - 1}{2} = 364$$

cubic fields of the 19-digit discriminant

$$D = -3161659186633662283$$

in under 3 CPU minutes

Jacobson, Lee, S. and Williams, Int. J. Number Theory **11** (2015)

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- $\mathbb{C} \to \mathbb{F}_{q^2}((x^{-1}))$ or $\mathbb{F}_q((x^{-1/2}))$

Problems

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- Hasse count is wrong
- There are three types of quadratic fields

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- imaginary if $\deg(D)$ is odd
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- real if $\deg(D)$ is even and $\mathrm{sgn}(D)$ is a square in $\mathbb{F}_q$
  - infinite place of $\mathbb{F}_q(t)$ splits
- unusual if $\deg(D)$ is even and $\mathrm{sgn}(D)$ is a non-square in $\mathbb{F}_q$
  - infinite place of $\mathbb{F}_q(t)$ is inert – *no number field analogue!*

Decomposition at Infinity in $\mathbb{K}$

Let $\mathbb{K}$ be a cubic extension of $\mathbb{F}_q(t)$ of square-free discriminant $D \in \mathbb{F}_q[t]$

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Hasse count does not include the red cases.

The Berwick Map

As before, triples of conjugate cubic function fields are mapped onto pairs of 3-torsion ideal classes in $\mathbb{F}_q[t, \sqrt{D}]$.

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For $\mathbb{F}_q(t, \sqrt{-3D})$ imaginary or unusual:

bijection onto non-principal ideal classes
nothing maps to the principal class

For $\mathbb{F}_q(t, \sqrt{-3D})$ real:

3-to-1 onto non-principal ideal classes
1-to-1 onto to the principal class

Some Counting

Put

$$r = 3\text{-rank}\big(\mathrm{Cl}(\mathbb{Q}(\sqrt{D})\big)$$
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Connection between r and s (Lee 2007):

$|r - s| \leq 1$

If $r \neq s$, then the unusual quadratic field has the bigger 3-rank

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So what are these extra 3^r cubic fields?

Answer: they are the fields with one infinite place that are missing from the Hasse count. In the ☺ cases, there are no such fields.

Reduced Ideals

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(Almost) reduced ideals produce λ with small norm: $|N(\lambda)| \le |D|^{3/2}$

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Yields again a small trace.

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- Principal class: take reduced ideal closest to $\lceil R/3 + g/2 \rfloor$
- Non-principal classes: take ideals closest to d, $R/3 + d$, $2R/3 + d$ where $-g/2 \leq d < R/3 - g/2$ and d can be explicitly computed **using integer arithmetic only!**

Example — Different 3-Rank

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$f(x) = x^3 - S(t)x + T(t)$ with

#	$S(t)$	$T(t)$
1	$5t^3 + 10t + 4$	$4t^6 + t^5 + t^3 + 9t^2 + 6t + 4$
2	$10t^4 + 9t^3 + t^2 + 5t + 9$	$10t^6 + 8t^5 + 5t^3 + 5t^2 + 5t + 3$
3	$6t^4 + 4t^3 + 10t + 4$	$5t^6 + 4t^5 + 3t^4 + 5t^3 + 3t^2 + t + 7$
4	$9t^4 + 4t^3 + 6t^2 + 5t + 1$	$t^6 + 4t^5 + 8t^4 + 9t^3 + 4t^2 + 7t + 5$
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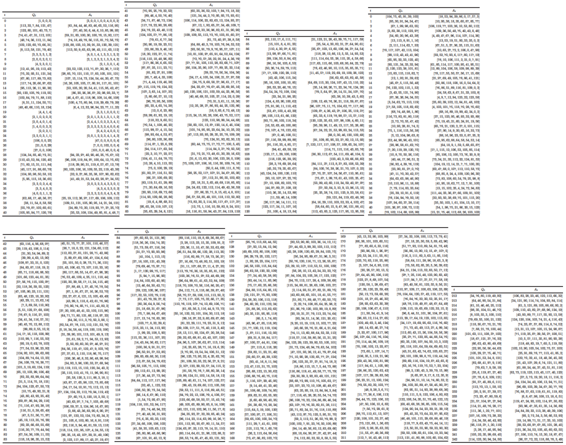

364 fields

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- Ideas can be extended to higher degree fields with quadratic resolvent fields

Thank You — Questions?